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On super-Jordanian $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ algebra*

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Abstract
A nonlinear realization of the nonstandard (super-Jordanian) version of $\mathcal{U}(sl(N|1))$ is given, for all N.

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1. Introduction

Jordanian and super-Jordanian quantum algebras have been recently used for several applications in physical problems. For instance, super-Jordanian $\mathcal{U}_{\mathsf{h}}(osp(1|2))$ algebra has been understood as the κ-deformation of the symmetry algebra of the super-conformal mechanics [1]. In another context, integrable deformed Hamiltonian systems have been introduced [2] via Poisson co-algebra associated with quantized Jordanian $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra. We believe that fully developed Hopf co-algebraic structure in a deformed basis for the $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ presented here will be useful in building and studying similar deformed fermionic integrable models. Furthermore, using the co-representation structure of the function algebra dually related to the universal enveloping algebra, a general method of constructing noncommutative (super)spaces has been recently developed [3] in the context of the quantum supergroup $OSp_q(1|2)$. Application of this method to the case of the dual quantum supergroup $SL_{\mathsf{h}}(N|1)$ will lead to new quantum superspaces inherently containing a *dimensional* deformation parameter. Influenced by these observations here we introduce super-Jordanian $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ algebra in the deformed basis set.

In a series of papers [4–7], we have proposed a new scheme which permits the construction of the nonstandard version $\mathcal{U}_{\mathsf{h}}(\mathfrak{g})$ of an enveloping (super)algebra $\mathcal{U}(\mathfrak{g})$ by a suitable contraction, from the corresponding standard ones $\mathcal{U}_q(\mathfrak{g})$. Our method hinges on

* In memory of our friend Professor Daniel Arnaudon.

obtaining the $\mathcal{R}_{\mathsf{h}}$-matrix, for all dimensions, of a (super)Jordanian quantum (super)algebra $\mathcal{U}_{\mathsf{h}}(\mathsf{g})$ from the $\mathcal{R}_q$-matrix associated with the standard quantum (super)algebra $\mathcal{U}_q(\mathsf{g})$ through a specific transformation G (singular in the $q \to 1$ limit), as follows:

$$\mathcal{R}_{\mathsf{h}} = \lim_{q\to 1}[\mathsf{G}^{-1} \otimes \mathsf{G}^{-1}]\mathcal{R}_q[\mathsf{G} \otimes \mathsf{G}], \tag{1.1}$$

where, for example, $\mathsf{G} = \mathsf{E}_q\left(\frac{\mathsf{h}\hat{e}_{1N}}{q-1}\right)$ for $\mathcal{U}_q\big(sl(N)\big)$ ($\hat{e}_{1N}$ is the longest positive root generator of $\mathcal{U}_q(sl(N))$) and $\mathsf{G} = \mathsf{E}_{q^2}\left(\frac{\mathsf{h}\hat{e}^2}{q^2-1}\right)$ for $\mathcal{U}_q(osp(2|1))$ ($\hat{e}$ is the fermionic positive simple root generator of $\mathcal{U}_q(osp(1|2))$). The deformed exponential map E_q is defined by

$$\begin{aligned} &\mathsf{E}_q(\eta) = \sum_{n=0}^{\infty} \frac{(\eta)^n}{[n]_q!}, \qquad && [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \\ &[n]_q! = [n]_q \times [n-1]_q!, \qquad && [0]_q! = 1. \end{aligned} \tag{1.2}$$

For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. This procedure yields a nonstandard deformation along with a nonlinear map of the h-Borel subalgebra on the corresponding classical Borel subalgebra, which can be artfully extended to the whole (super)algebra. The Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(N))$ arising from the process cited above corresponds to the classical matrix $r = h_{1N} \wedge e_{1N}$. Therefore, the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the full $\mathcal{U}_{\mathsf{h}}(sl(N))$ Hopf algebra, obtained, coincides with the universal $\mathcal{R}_{\mathsf{h}}$-matrix of the $\mathcal{U}_{\mathsf{h}}(sl(2))$ Hopf subalgebra [19] associated with the highest roots. In the case of $\mathcal{U}(osp(1|2))$,[5] the super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(osp(2|1))$ occurred from our treatment is associated with the classical matrix $r = h \wedge e^2 - e \wedge e$. The advantages of our technique are: (1) with an appropriate choice of basis, the Jordanian quantum Hopf (super)algebra, obtained by our process, can be endowed with a relatively simpler co-algebraic structure, and (2) our nonlinear map permits immediate explicit construction of the finite-dimensional irreducible representations.

Let us mention that in general, nonstandard quantum algebras are obtained by applying Drinfeld twist [8] to the corresponding Lie algebras (see [9–15] and references therein). The twist deformation of super-algebras was also discussed in the literature: [1, 16] ($\mathcal{U}(osp(1|2))$ case), [17] ($\mathcal{U}(osp(1|4))$ case) and [18] (general super-algebra case). We will not consider this way here.

The main object of this paper is to present how our contraction procedure works for $\mathcal{U}(sl(N|1))$ super-algebra for obtaining the nonstandard version $\mathcal{U}_{\mathsf{h}}(sl(N|1))$. For simplicity, we will limit here ourselves to $\mathcal{U}(sl(2|1))$ and $\mathcal{U}(sl(3|1))$. The construction of higher dimensional super-algebras $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ is presented, briefly, at the end of this paper. The manuscript is organized as follows: the super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ is introduced via a nonlinear map and proved to be a Hopf algebra. Higher dimensional super-algebras $\mathcal{U}_{\mathsf{h}}(sl(N|1))$, $N \geqslant 3$, are presented in sections 3 and 4. We conclude in section 5.

2. $\mathcal{U}_{\mathsf{h}}(sl(2|1))$: contraction, nonlinear map and Hopf structure

Let us recall the more important points concerning $sl(2|1)$: Let $A = (a_{ij})$ be the 2×2 matrix given by $a_{11} = 2, a_{12} = a_{21} = -1$ and $a_{22} = 0$. The Lie–Hopf super-algebra $\mathcal{U}(sl(2|1))$ is generated by the generators h_i, e_i and $f_i, i = 1, 2$, where h_1, h_2, e_1 and

[5] The recent work shows that there exist three distinct bialgebra structure on $osp(1|2)$ and all of them are co-boundary. We therefore have three distinct quantization of $osp(1|2)$.

f_1 are even ($\deg(h_1) = \deg(h_2) = \deg(e_1) = \deg(f_1) = 0$), while e_2 and f_2, are odd ($\deg(e_2) = \deg(f_2) = 1$), and the commutation relations

$$\begin{aligned} &[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \qquad [e_i, f_j] = \delta_{ij} h_i, \\ &[e_2, e_2] = [f_2, f_2] = 0, \qquad [e_1, [e_1, e_2]] = [f_1, [f_1, f_2]] = 0. \end{aligned} \tag{2.1}$$

The last two equations are called the Serre relations. The commutator [,] is understood as the $\mathbb{Z}_2$-graded one: $[a, b] = ab - (-)^{\deg(a)\deg(b)} ba$. Defining

$$e_3 = e_1 e_2 - e_2 e_1, \qquad f_3 = f_2 f_1 - f_1 f_2, \tag{2.2}$$

we obtain

$$\begin{aligned} &[e_1, e_3] = 0, \qquad [f_3, f_1] = 0, \qquad [e_2, e_3] = 0, \qquad [f_2, f_3] = 0, \\ &e_3^2 = f_3^2 = 0, \qquad [e_3, f_3] = h_1 + h_2 \equiv h_3, \qquad [f_1, e_3] = e_2, \text{etc.} \end{aligned} \tag{2.3}$$

Let us mention that there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathcal{U}(sl(2|1))$ such that

$$\phi : (h_1, h_2, h_3, e_1, e_2, e_3, f_1, f_2, f_3) \to (h_1, -h_3, -h_2, e_1, f_3, -f_2, f_1, -e_3, e_2). \tag{2.4}$$

The quasitriangular quantum Hopf super-algebra $\mathcal{U}_q(sl(2|1))$ (q is an arbitrary complex number), by analogy with $\mathcal{U}(sl(2|1))$, is generated by six elements $\hat{h}_i, \hat{e}_i$ and $\hat{f}_i, i = 1, 2$, under the relations

$$\begin{aligned} &[\hat{h}_i, \hat{h}_j] = 0, \qquad [\hat{h}_i, \hat{e}_j] = a_{ij}\hat{e}_j, \qquad [\hat{h}_i, \hat{f}_j] = -a_{ij}\hat{f}_j, \\ &[\hat{e}_i, \hat{f}_j] = \delta_{ij}\frac{q^{\hat{h}_i} - q^{-\hat{h}_i}}{q - q^{-1}}, \qquad \hat{e}_2^2 = \hat{f}_2^2 = 0, \\ &\hat{e}_1^2\hat{e}_2 - (q + q^{-1})\hat{e}_1\hat{e}_2\hat{e}_1 + \hat{e}_2\hat{e}_1^2 = \hat{f}_1^2\hat{f}_2 - (q + q^{-1})\hat{f}_1\hat{f}_2\hat{f}_1 + \hat{f}_2\hat{f}_1^2 = 0. \end{aligned} \tag{2.5}$$

All generators are even except for $\hat{e}_2$ and $\hat{f}_2$ which are odd and $\deg(\hat{h}_1) = \deg(\hat{h}_2) = \deg(\hat{e}_1) = \deg(\hat{f}_1) = 0$. The co-products, counits and antipodes are given by

$$\begin{aligned} &\Delta(\hat{e}_i) = \hat{e}_i \otimes q^{\hat{h}_i/2} + q^{-\hat{h}_i/2} \otimes \hat{e}_i, \qquad \epsilon(\hat{e}_i) = 0, \qquad S(\hat{e}_i) = -q^{\hat{h}_i/2}\hat{e}_i q^{-\hat{h}_i/2}, \\ &\Delta(\hat{f}_i) = \hat{f}_i \otimes q^{\hat{h}_i/2} + q^{-\hat{h}_i/2} \otimes \hat{f}_i, \qquad \epsilon(\hat{f}_i) = 0, \qquad S(\hat{f}_i) = -q^{\hat{h}_i/2}\hat{f}_i q^{-\hat{h}_i/2}, \\ &\Delta(\hat{h}_i) = \hat{h}_i \otimes 1 + 1 \otimes \hat{h}_i, \qquad \epsilon(\hat{h}_i) = 0, \qquad S(\hat{h}_i) = -\hat{h}_i. \end{aligned} \tag{2.6}$$

The universal $\mathcal{R}$-matrix is given in [20, 21]. Note that the definition of the Hopf super-algebra differs from that of the usual Hopf algebra by the supercommutativity of tensor product, i.e. $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)}(ac \otimes bd)$. For later use, we note that the fundamental representation of (2.5) is spanned by

$$\begin{aligned} &\hat{h}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{f}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\hat{h}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{f}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.7}$$

2.1. Contraction process

Following [4], the $\mathcal{R}_{\mathsf{h}}$ (h is an arbitrary complex number) matrix of the super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(sl(2|1))$, for arbitrary representations in the two tensor product sectors, can be also obtained from the $\mathcal{R}_q$-matrix associated with the Drinfeld–Jimbo quantum super-algebra $\mathcal{U}_q(sl(2|1))$ through a specific contraction. For simplicity and brevity, let us start with (fundamental irrep.) ⊗ (fundamental irrep.). The $\mathcal{R}_q$-matrix of $\mathcal{U}_q(sl(2|1))$ super-algebra in the (fund.) ⊗ (fund.) representation reads

$$R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{fund.})} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q-q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q-q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q-q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-2} \end{pmatrix}. \tag{2.8}$$

The $\mathcal{R}_{\mathsf{h}}$-matrix in the (fund. ⊗ fund.) representation is obtained, from (2.9), in the following manner:

$$R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{fund.})}$$
$$= \lim_{q\to 1}\left[\mathsf{E}_q^{-1}\left(\frac{\mathsf{h}\hat{e}_1}{q-1}\right)_{\text{fund.}} \otimes \mathsf{E}_q^{-1}\left(\frac{\mathsf{h}\hat{e}_1}{q-1}\right)_{\text{fund.}}\right] R_q \left[\mathsf{E}_q\left(\frac{\mathsf{h}\hat{e}_1}{q-1}\right)_{\text{fund.}} \otimes \mathsf{E}_q\left(\frac{\mathsf{h}\hat{e}_1}{q-1}\right)_{\text{fund.}}\right]$$
$$= \begin{pmatrix} 1 & \mathsf{h} & 0 & -\mathsf{h} & \mathsf{h}^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathsf{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\mathsf{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.9}$$

Similarly, using a Maple program[6] we obtain, for (fundamental irrep.) ⊗ (arbitrary irrep.), the following expression:

$$L \equiv R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{arb.})} = \begin{pmatrix} T & -\mathsf{h}H_1 + \frac{\mathsf{h}}{2}(T - T^{-1}) & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & (-1)^F \end{pmatrix}, \tag{2.10}$$

where

$$H_1 = \tfrac{1}{2}(T + T^{-1})h_1 = \sqrt{1 + \mathsf{h}^2 e_1^2}\, h_1, \qquad T^{\pm 1} = \pm \mathsf{h}e_1 + \sqrt{1 + \mathsf{h}^2 e_1^2}. \tag{2.11}$$

The above L operator allows immediate construction of the full Hopf structure of the Borel subalgebra of the $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ algebra via the FRT formalism[7].

2.2. Nonlinear map and Hopf structure

Following [4, 6], let us introduce the generator

$$F_1 = f_1 - \frac{\mathsf{h}^2}{4} e_1\big(h_1^2 - 1\big). \tag{2.12}$$

[6] Our program was performed for (fund.) ⊗ (fund.), (fund.) ⊗ (vect.) etc.

[7] The algebraic and co-algebraic properties of the Borel subalgebra are respectively given by $R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{fund.})} L_1 L_2 = L_2 L_1 R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{fund.})}$, $\Delta(L) = L\dot{\otimes}L$, $\varepsilon(L) = 1$ and $S(L) = L^{-1}$.

We then show that

$$
\begin{aligned}
&TT^{-1} = T^{-1}T = 1, \qquad [H_1, T^{\pm 1}] = T^{\pm 2} - 1,\\
&[T^{\pm 1}, F_1] = \pm\frac{\mathsf{h}}{2}(H_1T^{\pm 1} + T^{\pm 1}H_1),\\
&[H_1, F_1] = -\frac{1}{2}(TF_1 + F_1T + T^{-1}F_1 + F_1T^{-1}),
\end{aligned} \tag{2.13}
$$

with the well-known co-products, counits and antipodes [22]

$$
\begin{aligned}
&\Delta(H_1) = H_1 \otimes T + T^{-1} \otimes H_1, &&\Delta(T^{\pm 1}) = T^{\pm 1} \otimes T^{\pm 1},\\
&\Delta(F_1) = F_1 \otimes T + T^{-1} \otimes F_1, &&S(H_1) = -TH_1T^{-1},\\
&S(T^{\pm 1}) = T^{\mp 1}, &&S(F_1) = -TF_1T^{-1},\\
&\epsilon(H_1) = \epsilon(F_1) = 0, &&\epsilon(T^{\pm 1}) = 1.
\end{aligned} \tag{2.14}
$$

This implies that Ohn's structure follows from the bosonic generators $\{h_1, e_1, f_1\}$. The algebraic properties (2.11) and (2.12) exhibit clearly the embedding of $\mathcal{U}_{\mathsf{h}}(sl(2))$ in $\mathcal{U}_{\mathsf{h}}(sl(2|1))$.

Now to complete the $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ super-algebra, we introduce the following h-deformed fermionic root generators:

$$
\begin{aligned}
&H_2 = h_2 - \frac{\mathsf{h}^2}{2}e_1^2h_1, \qquad E_2 = e_2 - \frac{\mathsf{h}^2}{4}e_1e_3(2h_1 + 1), \qquad F_2 = f_2,\\
&H_3 = h_3 + \frac{\mathsf{h}^2}{2}e_1^2h_1, \qquad E_3 = e_3, \qquad F_3 = f_3 + \frac{\mathsf{h}^2}{4}e_1f_2(2h_1 + 1).
\end{aligned} \tag{2.15}
$$

The generators E_2, E_3, F_2 and F_3 are odd, while H_2 and H_3 are even. The expressions (2.11), (2.12) and (2.15) define a realization of the super-Jordanian subalgebra $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ with the classical generators via a nonlinear map (other invertible maps relating the super-Jordanian and the classical generators may also be considered) and permit immediate explicit construction of the finite-dimensional irreducible representations of the $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ super-algebra. In the following we quote only our final results:

Proposition 1. *The nonstandard (super-Jordanian) enveloping super-algebra $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ is an associative super-algebra over $\mathbb{C}$ generated by $\{H_1, T, T^{-1}, F_1, H_2, E_2, F_2, H_3, E_3, F_3\}$ satisfying, along with (2.15) and (2.17), the commutation relations*

$$
\begin{aligned}
&[H_1, H_2] = -\frac{1}{4}(T - T^{-1})^2H_1, \qquad [H_1, H_3] = \frac{1}{4}(T - T^{-1})^2H_1, \qquad [H_2, H_3] = 0,\\
&[H_1, E_2] = -\frac{1}{2}(T + T^{-1})E_2 - \frac{\mathsf{h}}{2}(T - T^{-1})E_3H_1 - \frac{\mathsf{h}}{4}(T^2 - T^{-2})E_3,\\
&[H_1, F_3] = -\frac{1}{2}(T + T^{-1})F_3 + \frac{\mathsf{h}}{2}(T - T^{-1})F_2H_1 + \frac{\mathsf{h}}{4}(T^2 - T^{-2})F_2,\\
&[H_1, F_2] = \frac{1}{2}(T + T^{-1})F_2, \qquad [H_1, E_3] = \frac{1}{2}(T + T^{-1})E_3,\\
&[H_2, T^{\pm 1}] = -\frac{1}{4}(T^{\pm 3} - T^{\mp 1}), \qquad [H_3, T^{\pm 1}] = \frac{1}{4}(T^{\pm 3} - T^{\mp 1}),\\
&[H_2, F_1] = \frac{1}{4}(T+T^{-1})^2F_1 - \frac{\mathsf{h}}{4}(T - T^{-1})H_1^2 - \frac{\mathsf{h}}{4}(T^2-T^{-2})H_1 - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1}),\\
&[H_3, F_1] = -\frac{1}{4}(T + T^{-1})^2F_1 + \frac{\mathsf{h}}{4}(T - T^{-1})H_1^2 + \frac{\mathsf{h}}{4}(T^2 - T^{-2})H_1 + \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T+T^{-1}),\\
&[H_2, E_2] = \frac{\mathsf{h}}{16}(T + T^{-1})(T^2 - T^{-2})E_3 + \frac{1}{8}(T - T^{-1})^2E_2,
\end{aligned}
$$

$$
\begin{aligned}
&[H_3, F_3] = \frac{\mathsf{h}}{16}(T - T^{-1})(T^2 - T^{-2})F_2 - \frac{1}{8}(T - T^{-1})^2 F_3,\\
&[H_2, F_3] = \frac{1}{8}(T^2 + 6 + T^{-2})F_3 - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1})F_2,\\
&[H_3, E_2] = -\frac{1}{8}(T^2 + 6 + T^{-2})E_2 - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1})E_3,\\
&[H_2, F_2] = -\frac{1}{8}(T - T^{-1})^2 F_2, \qquad [H_3, E_3] = \frac{1}{8}(T - T^{-1})^2 E_3,\\
&[H_3, F_2] = \frac{1}{8}(T^2 + 6 + T^{-2})F_2, \qquad [H_2, E_3] = -\frac{1}{8}(T^2 + 6 + T^{-2})E_3,\\
&[E_2, F_2] = H_2 - \frac{1}{16}(T - T^{-1})^2 - \frac{\mathsf{h}}{4}(T - T^{-1})E_3F_2,\\
&[E_3, F_3] = H_3 + \frac{1}{16}(T - T^{-1})^2 + \frac{\mathsf{h}}{4}(T - T^{-1})F_2E_3,\\
&[T^{\pm 1}, F_2] = [T^{\pm 1}, E_3] = 0, \qquad F_2^2 = E_3^2 = 0, \qquad [F_2, F_1] = F_3, \qquad [F_1, E_3] = E_2,\\
&E_2^2 = \frac{\mathsf{h}}{4}(T - T^{-1})E_3E_2, \qquad F_3^2 = -\frac{\mathsf{h}}{4}(T - T^{-1})F_2F_3, \qquad [E_2, E_3] = [F_2, F_3] = 0,\\
&[T^{\pm 1}, E_2] = \pm\frac{\mathsf{h}}{2}(T^{\pm 2} + 1)E_3, \quad [T^{\pm 1}, F_3] = \mp\frac{\mathsf{h}}{2}(T^{\pm 2} + 1)F_2, \quad [F_2, E_3] = \frac{1}{2\mathsf{h}}(T - T^{-1}),\\
&[E_2, F_1] = \frac{\mathsf{h}}{4}(T - T^{-1})E_2 + \frac{\mathsf{h}}{2}(T - T^{-1})E_3F_1 - \frac{\mathsf{h}^2}{4}E_3H_1^2\\
&\qquad - \frac{3\mathsf{h}^2}{8}(T + T^{-1})E_3H_1 - \frac{\mathsf{h}^2}{2}E_3 - \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2E_3,\\
&[F_3, F_1] = \frac{\mathsf{h}}{4}(T - T^{-1})F_3 - \frac{\mathsf{h}}{2}(T - T^{-1})F_2F_1 + \frac{\mathsf{h}^2}{4}F_2H_1^2 + \frac{3\mathsf{h}^2}{8}(T + T^{-1})F_2H_1 + \frac{\mathsf{h}^2}{2}F_2\\
&\qquad + \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2F_2,\\
&[F_3, E_2] = F_1 - \frac{\mathsf{h}}{4}(T - T^{-1})F_2E_2 + \frac{\mathsf{h}}{4}(T - T^{-1})E_3F_3 - \frac{\mathsf{h}}{8}(T - T^{-1})H_1^2 - \frac{\mathsf{h}}{8}(T^2 - T^{-2})H_1\\
&\qquad - \frac{\mathsf{h}}{16}H_1(T^2 - T^{-2}) - \frac{7\mathsf{h}}{128}(T - T^{-1})^3.
\end{aligned} \tag{2.16}
$$

The $\mathbb{Z}_2$-grading in $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ is uniquely defined by the requirement that the only odd generators are E_2, F_2, E_3 and F_3. It is obvious that as $\mathsf{h} \to 0$, we have $(E_2, F_2, H_2, E_3, F_3, H_3) \to (e_2, f_2, h_2, e_3, f_3, h_3)$.

Proposition 2. *Let us note that there exists a $\mathbb{C}$-algebra automorphism of $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ such that*

$$
\Phi(T^{\pm 1}, F_1, H_1, E_2, F_2, H_2, E_3, F_3, H_3) \longrightarrow (T^{\pm 1}, F_1, H_1, F_3, -E_3, -H_3, -F_2, E_2, -H_2). \tag{2.17}
$$

(For $\mathsf{h} = 0$, this automorphism reduces to (2.5)).

Proposition 3. *The nonstandard (super-Jordanian) quantum enveloping super-algebra $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ admits a Hopf structure with co-products, antipodes and counits determined* *by (2.15) and*

$$\begin{aligned}
\Delta(E_2) &= E_2 \otimes T^{1/2} + T^{-1/2} \otimes E_2 + \frac{\mathsf{h}}{4} T^{-1} E_3 \otimes (T^{-1/2} H_1 + H_1 T^{-1/2}) \\
&\quad - \frac{\mathsf{h}}{4}(T^{1/2} H_1 + H_1 T^{1/2}) \otimes T E_3, \\
\Delta(F_2) &= F_2 \otimes T^{-1/2} + T^{1/2} \otimes F_2, \\
\Delta(E_3) &= E_3 \otimes T^{-1/2} + T^{1/2} \otimes E_3, \\
\Delta(F_3) &= F_3 \otimes T^{1/2} + T^{-1/2} \otimes F_3 - \frac{\mathsf{h}}{4} T^{-1} F_2 \otimes (T^{-1/2} H_1 + H_1 T^{-1/2}) \\
&\quad + \frac{\mathsf{h}}{4}(T^{1/2} H_1 + H_1 T^{1/2}) \otimes T F_2, \\
\Delta(H_2) &= H_2 \otimes 1 + 1 \otimes H_2 + \frac{1}{4} T H_1 \otimes (1 - T^2) + \frac{1}{4}(1 - T^{-2}) \otimes T^{-1} H_1, \\
\Delta(H_3) &= H_3 \otimes 1 + 1 \otimes H_3 - \frac{1}{4} T H_1 \otimes (1 - T^2) - \frac{1}{4}(1 - T^{-2}) \otimes T^{-1} H_1, \\
S(E_2) &= -E_2 - \frac{\mathsf{h}}{2}(T + T^{-1}) E_3, \qquad S(F_3) = -F_3 + \frac{\mathsf{h}}{2}(T + T^{-1}) F_2, \\
S(F_2) &= -F_2, \qquad S(E_3) = -E_3, \\
S(H_2) &= -H_2 + \frac{1}{2}(T^{-2} - 1), \qquad S(H_3) = -H_3 - \frac{1}{2}(T^{-2} - 1), \\
\epsilon(H_2) &= \epsilon(H_3) = \epsilon(E_2) = \epsilon(F_2) = \epsilon(E_3) = \epsilon(F_3) = 0.
\end{aligned} \tag{2.18}$$

All the Hopf super-algebra axioms can be verified by direct calculations. We remark that our co-products have simpler forms compared to those given in the literature [9–18]. This is one main advantage of our procedure.

Proposition 4. *The universal $\mathcal{R}_{\mathsf{h}}$-matrix of $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ has the following form:*

$$\mathcal{R}_{\mathsf{h}} = \exp(-\mathsf{h} X_1 \otimes T H_1) \exp(\mathsf{h} T H_1 \otimes X_1), \tag{2.19}$$

where $X_1 = \mathsf{h}^{-1} \ln T$. Element (2.19) coincides with the pure $\mathcal{U}_{\mathsf{h}}(sl(2))$ universal $\mathcal{R}_{\mathsf{h}}$-matrix [19].

3. $\mathcal{U}(sl(3|1))$: Nonstandard quantization and Hopf structure

The major interest of our approach is that it can be generalized for obtaining super-Jordanian quantum super-algebras $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ of higher dimensions. We start here with $\mathcal{U}_{\mathsf{h}}(sl(3|1))$. In our notations e_{ij} is an $(N+1) \times (N+1)$ matrix with only the (i, j) matrix element being equal to 1, all other matrix elements are zero. Let $h_{12} = e_{11} - e_{22}$, $h_{23} = e_{22} - e_{33}, h_{34} = e_{33} + e_{44}, e_{12}, e_{23}, e_{34}, e_{21}, e_{32}$ and e_{43} be the standard Chevalley generators of $\mathcal{U}(sl(3|1))$. The generators $h_{12}, h_{23}, e_{12}, e_{23}, e_{21}, e_{32}$, and h_{34} are even, while e_{34} and e_{43} are odd. The generators corresponding to other roots, obtained by the action of the Weyl group, are denoted by $e_{13} = [e_{12}, e_{23}]$, $e_{14} = [e_{13}, e_{34}]$, $e_{24} = [e_{23}, e_{34}]$, $e_{31} = [e_{32}, e_{21}]$, $e_{41} = [e_{43}, e_{31}]$, $e_{42} = [e_{43}, e_{32}]$, $h_{13} = e_{11} - e_{33} \equiv h_{12} + h_{23}$, $h_{14} = e_{11} + e_{44} \equiv h_{13} + h_{34}$ and $h_{24} = e_{22} + e_{44} \equiv h_{23} + h_{34}$.[8] The commutator [,] is understood as the $\mathbb{Z}_2$-graded one, i.e.

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-)^{\deg(e_{ij}) \deg(e_{kl})} \delta_{li} e_{kj}. \tag{3.1}$$

[8] The elements $\{h_{12}, h_{23}, e_{12}, e_{23}, e_{21}, e_{32}, e_{13}, e_{31}, h_{13}\}$ build here the subalgebra $\mathcal{U}(sl(3))$ of $\mathcal{U}(sl(3|1))$.

There exists a $\mathbb{C}$-algebra automorphism ϕ of $\mathcal{U}(sl(3|1))$ such that

$$\phi(e_{12}, e_{21}, h_{12}, e_{23}, e_{32}, h_{23}, e_{34}, e_{43}, h_{34}, \ldots) \longrightarrow (e_{23}, e_{32}, h_{23}, e_{12}, e_{12}, h_{12}, e_{41}, e_{14}, -h_{14}, \ldots). \tag{3.2}$$

3.1. The bosonic part: $\mathcal{U}_{\mathsf{h}}(sl(3))$ subalgebra

As in the $\mathcal{U}_{\mathsf{h}}(sl(2|1))$ super-algebra, the super-Jordanian deformation arises here from the bosonic generators corresponding to the higher root, i.e. e_{13}, e_{31} and h_{13}. These generators are deformed as follows[9]:

$$T^{\pm 1} = \pm \mathsf{h} e_{13} + \sqrt{1 + \mathsf{h}^2 e_{13}^2}, \qquad H_{13} = \sqrt{1 + \mathsf{h}^2 e_{13}^2} h_{13}, \qquad E_{31} = e_{31} - \frac{\mathsf{h}^2}{4} e_{13}\left(h_{13}^2 - 1\right). \tag{3.3}$$

First to complete the $\mathcal{U}_{\mathsf{h}}(sl(3)) \subset \mathcal{U}_{\mathsf{h}}(sl(3|1))$ subalgebra (*the bosonic part of* $\mathcal{U}_{\mathsf{h}}(sl(3|1))$), let us introduce the following h-deformed generators:

$$\begin{aligned} H_{12} &= h_{12} + \frac{\mathsf{h}^2}{2} e_{13}^2 h_{13}, \qquad E_{12} = e_{12}, \qquad E_{21} = e_{21} + \frac{\mathsf{h}^2}{4} e_{23} e_{13} (2h_{13} + 1), \\ H_{23} &= h_{23} + \frac{\mathsf{h}^2}{2} e_{13}^2 h_{13}, \qquad E_{23} = e_{23}, \qquad E_{32} = e_{32} - \frac{\mathsf{h}^2}{4} e_{12} e_{13} (2h_{13} + 1), \end{aligned} \tag{3.4}$$

where it is obvious that as $\mathsf{h} \to 0$, we have $(H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, ; H_{13}, T, T^{-1}, E_{31}) \to (h_{12}, e_{12}, e_{21}, h_{23}, e_{23}, e_{32}, h_{13}, 1, 1, e_{31})$. Expressions (3.3) and (3.4) define a realization of the Jordanian subalgebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ embedded in $\mathcal{U}_{\mathsf{h}}(sl(3|1))$ with the classical generators via a nonlinear map. Another map has been considered in [6]. Our construction leads to the following results.

Proposition 5. *The generating elements* $\{H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, H_{13}, T, T^{-1}, E_{31}\}$ *of the Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(3))$ *obey the following commutations rules:*

$$TT^{-1} = T^{-1}T = 1, \qquad [H_{13}, T^{\pm 1}] = T^{\pm 2} - 1, \qquad [T^{\pm 1}, E_{31}] = \pm \frac{\mathsf{h}}{2}(H_{13}T^{\pm 1} + T^{\pm 1}H_{13}),$$

$$[H_{13}, E_{31}] = -\frac{1}{2}((T + T^{-1})E_{31} + E_{31}(T + T^{-1})), \qquad [H_{12}, H_{23}] = 0,$$

$$[H_{12}, H_{13}] = -\frac{1}{4}(T - T^{-1})^2 H_{13}, \qquad [H_{23}, H_{13}] = -\frac{1}{4}(T - T^{-1})^2 H_{13},$$

$$[H_{12}, E_{12}] = 2E_{12} + \frac{1}{8}(T - T^{-1})^2 E_{12}, \qquad [H_{12}, E_{23}] = -E_{23} + \frac{1}{8}(T - T^{-1})^2 E_{23},$$

$$[H_{23}, E_{12}] = -E_{12} + \frac{1}{8}(T - T^{-1})^2 E_{12}, \qquad [H_{23}, E_{23}] = 2E_{23} + \frac{1}{8}(T - T^{-1})^2 E_{23},$$

[9] Similar to [6], by applying the contraction process on the R_q-matrix in the (fund.$\otimes$arb.), associated with $\mathcal{U}_q(sl(3|1))$, we obtain

$$R_{\mathsf{h}}|_{(\text{fund.}\otimes\text{arb.})} = \begin{pmatrix} T & 2\mathsf{h}T^{-1/2}e_{23} & -\frac{\mathsf{h}}{2}(T + T^{-1})(h_1 + h_2) + \frac{\mathsf{h}}{2}(T - T^{-1}) & 0 \\ 0 & I & -2\mathsf{h}T^{1/2}e_{12} & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & (-1)^F \end{pmatrix}.$$

$$
\begin{aligned}
[H_{12}, E_{21}] &= -2E_{21} - \frac{1}{8}(T - T^{-1})^2 E_{21} + \frac{\mathsf{h}}{16}(T + T^{-1})(T^2 - T^{-2})E_{23},\\
[H_{23}, E_{32}] &= -2E_{32} - \frac{1}{8}(T - T^{-1})^2 E_{32} - \frac{\mathsf{h}}{16}(T + T^{-1})(T^2 - T^{-2})E_{12},\\
[H_{12}, E_{32}] &= E_{32} - \frac{1}{8}(T - T^{-1})^2 E_{32} - \frac{\mathsf{h}}{16}(T + T^{-1})(T^2 - T^{-2})E_{12},\\
[H_{23}, E_{21}] &= E_{21} - \frac{1}{8}(T - T^{-1})^2 E_{21} + \frac{\mathsf{h}}{16}(T + T^{-1})(T^2 - T^{-2})E_{23},\\
[H_{13}, E_{12}] &= \frac{1}{2}(T + T^{-1})E_{12}, \qquad [H_{13}, E_{23}] = \frac{1}{2}(T + T^{-1})E_{23},\\
[H_{13}, E_{21}] &= -\frac{1}{2}(T + T^{-1})E_{21} + \frac{\mathsf{h}}{2}(T - T^{-1})E_{23}H_{13} + \frac{\mathsf{h}}{4}(T^2 - T^{-2})E_{23},\\
[H_{13}, E_{32}] &= -\frac{1}{2}(T + T^{-1})E_{32} - \frac{\mathsf{h}}{2}(T - T^{-1})E_{12}H_{13} - \frac{\mathsf{h}}{4}(T^2 - T^{-2})E_{12},\\
[E_{21}, F_{31}] &= \frac{\mathsf{h}}{4}(T - T^{-1})E_{21} - \frac{\mathsf{h}}{2}(T - T^{-1})E_{23}E_{31} + \frac{\mathsf{h}^2}{4}E_{23}H_{13}^2 + \frac{3\mathsf{h}^2}{8}(T + T^{-1})E_{23}H_{13}\\
&\quad + \frac{\mathsf{h}^2}{2}E_{23} + \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2 E_{23},\\
[E_{32}, F_{31}] &= \frac{\mathsf{h}}{4}(T - T^{-1})E_{32} + \frac{\mathsf{h}}{2}(T - T^{-1})E_{12}E_{31} - \frac{\mathsf{h}^2}{4}E_{12}H_{13}^2 - \frac{3\mathsf{h}^2}{8}(T + T^{-1})E_{12}H_{13}\\
&\quad - \frac{\mathsf{h}^2}{2}E_{12} - \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2 E_{12},\\
[H_{12}, T^{\pm 1}] &= -\frac{1}{4}(T^{\pm 3} - T^{\mp 1}), \qquad [H_{23}, T^{\pm 1}] = -\frac{1}{4}(T^{\pm 3} - T^{\mp 1}),\\
[H_{12}, E_{31}] &= -\frac{1}{4}(T + T^{-1})^2 E_{31} + \frac{\mathsf{h}}{4}(T - T^{-1})H_{13}^2 + \frac{\mathsf{h}}{2}(T + T^{-1})E_{23}H_{13}\\
&\quad + \frac{\mathsf{h}}{16}(T^3 + T - T^{-1} - T^{-3}),\\
[H_{23}, E_{31}] &= -\frac{1}{4}(T + T^{-1})^2 E_{31} + \frac{\mathsf{h}}{4}(T - T^{-1})H_{13}^2 - \frac{\mathsf{h}}{2}(T + T^{-1})E_{12}H_{13}\\
&\quad + \frac{\mathsf{h}}{16}(T^3 + T - T^{-1} - T^{-3}),\\
[F_{32}, E_{21}] &= F_{31} + \frac{\mathsf{h}}{4}(T - T^{-1})(E_{12}E_{21} + E_{23}E_{32}) - \frac{\mathsf{h}}{8}(T - T^{-1})H_{13}^2 - \frac{\mathsf{h}}{4}(T - T^{-1})\\
&\quad - \frac{3\mathsf{h}}{16}(T^2 - T^{-2})H_{13} - \frac{9\mathsf{h}}{128}(T - T^{-1})^3,\\
[E_{12}, E_{21}] &= H_{12} + \frac{1}{16}(T - T^{-1})^2 - \frac{\mathsf{h}}{4}(T - T^{-1})E_{23}E_{12},\\
[E_{23}, E_{32}] &= H_{23} + \frac{1}{16}(T - T^{-1})^2 + \frac{\mathsf{h}}{4}(T - T^{-1})E_{12}E_{23}, \quad [T^{\pm 1}, E_{12}] = [T^{\pm 1}, E_{23}] = 0,\\
[E_{23}, E_{21}] &= -\frac{\mathsf{h}}{4}(T - T^{-1})E_{23}^2, \qquad [E_{12}, E_{32}] = \frac{\mathsf{h}}{4}(T - T^{-1})E_{12}^2,\\
[T^{\pm 1}, E_{21}] &= \mp\frac{\mathsf{h}}{2}(T^{\pm 2} + 1)E_{23}, \qquad [T^{\pm 1}, E_{32}] = \pm\frac{\mathsf{h}}{2}(T^{\pm 2} + 1)E_{12},\\
[E_{12}, E_{23}] &= \frac{1}{2\mathsf{h}}(T - T^{-1}).
\end{aligned} \tag{3.5}
$$

The other commutators remain undeformed.

3.2. The fermionic part

To describe the fermionic part of the super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(sl(3|1))$, we define the elements

$$
\begin{aligned}
&H_{34} = h_{34} - \frac{\mathsf{h}^2}{2} e_{13}^2 h_{13}, \qquad E_{34} = e_{34} - \frac{\mathsf{h}^2}{4} e_{13} e_{14}(2h_{13}+1), \qquad E_{43} = e_{43},\\
&H_{24} = h_{24}, \qquad E_{24} = e_{24}, \qquad E_{42} = e_{42}, \qquad\qquad (3.6)\\
&H_{14} = h_{14} + \frac{\mathsf{h}^2}{2} e_{13}^2 h_{13}, \qquad E_{14} = e_{14}, \qquad E_{41} = e_{41} + \frac{\mathsf{h}^2}{4} e_{13} e_{43}(2h_{13}+1).
\end{aligned}
$$

The generators E_{34}, E_{43}, E_{24}, E_{42}, E_{14} and E_{41} are odd, while H_{34}, H_{24} and H_{14} are even. Expressions (3.3), (3.4) and (3.6) constitute a nonlinear realization of the super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(sl(3|1))$ with the classical generators. Let us remark that the $\mathbb{C}$-algebra automorphism (3.2) can be easily extended to our construction, i.e.

$$
\begin{aligned}
&\phi(E_{12}, E_{21}, H_{12}, E_{23}, E_{32}, H_{23}, E_{34}, E_{43}, H_{34}, \ldots)\\
&\qquad \longrightarrow (E_{23}, E_{32}, H_{23}, E_{12}, E_{12}, H_{12}, E_{41}, E_{14}, -H_{14}, \ldots). \qquad (3.7)
\end{aligned}
$$

Proposition 6. *The super-Jordanian quantum super-algebra $\mathcal{U}_{\mathsf{h}}(sl(3|1))$ is then an associative super-algebra over $\mathbb{C}$ spanned by $\{H_{12}, E_{12}, E_{21}, H_{23}, E_{23}, E_{32}, H_{13}, T, T^{-1}, E_{31}, H_{34}, E_{34}, E_{43}, H_{24}, E_{24}, E_{42}, H_{14}, E_{14}, E_{41}\}$, satisfying along with (3.5), the commutation relations (we list here only the deformed commutator)*

$$
\begin{aligned}
&[H_{13}, H_{34}] = -\frac{1}{4}(T - T^{-1})^2 H_{13}, \qquad [H_{13}, H_{14}] = \frac{1}{4}(T - T^{-1})^2 H_{13},\\
&[H_{13}, E_{14}] = \frac{1}{2}(T + T^{-1}) E_{14}, \qquad [H_{13}, E_{43}] = \frac{1}{2}(T + T^{-1}) E_{43},\\
&[H_{13}, E_{41}] = -\frac{1}{2}(T + T^{-1}) E_{41} + \frac{\mathsf{h}}{2}(T - T^{-1}) E_{43} H_{13} + \frac{\mathsf{h}}{4}(T^2 - T^{-2}) E_{43},\\
&[H_{13}, E_{34}] = -\frac{1}{2}(T + T^{-1}) E_{34} - \frac{\mathsf{h}}{2}(T - T^{-1}) E_{14} H_{13} - \frac{\mathsf{h}}{2}(T^2 - T^{-2}) E_{14},\\
&[H_{34}, E_{14}] = -\left(1 + \frac{1}{8}(T - T^{-1})^2\right) E_{14}, \qquad [H_{14}, E_{43}] = \left(1 + \frac{1}{8}(T - T^{-1})^2\right) E_{43},\\
&[H_{34}, E_{41}] = \left(1 + \frac{1}{8}(T - T^{-1})^2\right) E_{41} - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1}) E_{43},\\
&[H_{34}, E_{34}] = \frac{1}{8}(T - T^{-1})^2 E_{34} + \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T - T^{-1}) E_{14},\\
&[H_{34}, E_{43}] = -\frac{1}{8}(T - T^{-1})^2 E_{43},\\
&[H_{34}, T^{\pm 1}] = -\frac{1}{4}(T^{\pm 3} - T^{\mp 1}), \qquad [H_{14}, T^{\pm 1}] = \frac{1}{4}(T^{\pm 3} - T^{\mp 1}),\\
&[H_{34}, E_{31}] = \frac{1}{4}(T + T^{-1})^2 E_{31} - \frac{\mathsf{h}}{4}(T - T^{-1}) H_{13}^2 - \frac{\mathsf{h}}{4}(T^2 - T^{-2}) H_{13}\\
&\qquad - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1}),\\
&[H_{14}, E_{31}] = -\frac{1}{4}(T + T^{-1})^2 E_{31} + \frac{\mathsf{h}}{4}(T - T^{-1}) H_{13}^2 + \frac{\mathsf{h}}{4}(T^2 - T^{-2}) H_{13}\\
&\qquad + \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1}),\\
&[H_{14}, E_{34}] = -\left(1 + \frac{1}{8}(T - T^{-1})^2\right) E_{34} - \frac{\mathsf{h}}{16}(T^2 - T^{-2})(T + T^{-1}) E_{43},
\end{aligned}
$$

$$[T^{\pm 1}, E_{34}] = \pm\frac{\mathsf{h}}{2}(T^{\pm 2}+1)E_{14}, \qquad [T^{\pm 1}, E_{41}] = \mp\frac{\mathsf{h}}{2}(T^{\pm 2}+1)E_{43},$$
$$[E_{43}, E_{14}] = \frac{1}{2\mathsf{h}}(T - T^{-1}),$$
$$[E_{34}, E_{43}] = H_{34} - \frac{1}{16}(T - T^{-1})^2 - \frac{\mathsf{h}}{4}(T - T^{-1})E_{14}E_{43},$$
$$[E_{14}, E_{41}] = H_{14} + \frac{1}{16}(T - T^{-1})^2 + \frac{\mathsf{h}}{4}(T - T^{-1})E_{43}E_{14},$$
$$[E_{43}, E_{31}] = \frac{\mathsf{h}}{4}(T - T^{-1})E_{34} + \frac{\mathsf{h}}{2}(T - T^{-1})E_{14}E_{31} - \frac{\mathsf{h}^2}{4}E_{14}H_{13}^2 - \frac{3\mathsf{h}^2}{8}(T + T^{-1})E_{14}H_{13} - \frac{\mathsf{h}^2}{2}E_{14} - \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2E_{14},$$
$$[E_{41}, E_{31}] = \frac{\mathsf{h}}{4}(T - T^{-1})E_{41} - \frac{\mathsf{h}}{2}(T - T^{-1})E_{43}E_{31} + \frac{\mathsf{h}^2}{4}E_{43}H_{13}^2 + \frac{3\mathsf{h}^2}{8}(T + T^{-1})E_{43}H_{13} + \frac{\mathsf{h}^2}{2}E_{43} + \frac{15\mathsf{h}^2}{64}(T - T^{-1})^2E_{43},$$
$$[E_{43}, E_{32}] = F_{42} + \frac{\mathsf{h}}{4}(T - T^{-1})E_{12}E_{43},$$
$$E_{34}^2 = \frac{\mathsf{h}}{4}(T - T^{-1})E_{14}E_{34}, \qquad E_{41}^2 = -\frac{\mathsf{h}}{4}(T - T^{-1})E_{43}E_{41},$$
$$[T^{\pm 1}, E_{14}] = 0, \qquad [T^{\pm 1}, E_{43}] = 0, \qquad [T^{\pm 1}, E_{24}] = 0, \qquad [T^{\pm 1}, E_{42}] = 0,$$
$$[E_{34}, E_{41}] = F_{31} - \frac{\mathsf{h}}{4}(T - T^{-1})E_{43}E_{34} + \frac{\mathsf{h}}{4}(T - T^{-1})E_{14}F_{41} - \frac{\mathsf{h}}{8}(T - T^{-1})H_{13}^2 - \frac{\mathsf{h}}{8}(T^2 - T^{-2})H_{13}^2 - \frac{\mathsf{h}}{16}H_{13}(T^2 - T^{-2}) + \frac{7\mathsf{h}}{128}(T - T^{-1})^3. \tag{3.8}$$

The coalgebraic structure will be presented, for the general case, in the following section.

4. $\mathcal{U}(sl(N|1))$: Generalization

From the above studies, it is easy to see that:

Proposition 7. *The super-algebra $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ can be realized via the nonlinear map:*

$$T^{\pm 1} = \pm\mathsf{h}e_{1N} + \sqrt{1 + \mathsf{h}^2e_{1N}^2}, \quad H_{1N} = \sqrt{1 + \mathsf{h}^2e_{1N}^2}h_{1N}, \quad E_{N1} = e_{N1} - \frac{\mathsf{h}^2}{4}e_{1N}\big(h_{1N}^2 - 1\big),$$
$$H_{ij} = h_{ij} + \frac{\mathsf{h}^2}{2}(\delta_{i1} + \delta_{jN})e_{1N}^2h_{1N}, \qquad i < j \in \{1, 2, \ldots, N\} \quad and \quad (i, j) \neq (1, N),$$
$$E_{ij} = e_{ij}, \qquad i < j \in \{1, 2, \ldots, N\} \quad and \quad (i, j) \neq (1, N),$$
$$E_{ji} = e_{ji} + \frac{\mathsf{h}^2}{4}(\delta_{i1}e_{jN} - \delta_{Nj}e_{1i})(2h_{1N} + 1), \quad i < j \in \{1, 2, \ldots, N\} \; and \; (i, j) \neq (1, N),$$
$$H_{i,N+1} = h_{i,N+1} + \frac{\mathsf{h}^2}{2}(\delta_{i1} - \delta_{iN})e_{1N}^2h_{1N}, \qquad i \in \{1, 2, \ldots, N\},$$
$$E_{i,N+1} = e_{i,N+1} - \frac{\mathsf{h}^2}{4}\delta_{iN}e_{1,N+1}e_{1N}(2h_{1N} + 1), \qquad i \in \{1, 2, \ldots, N\},$$
$$E_{N+1,i} = e_{N+1,i} + \frac{\mathsf{h}^2}{4}\delta_{iN}e_{N+1,N}e_{1N}(2h_{1N} + 1), \qquad i \in \{1, 2, \ldots, N\}, \tag{4.1}$$

with the co-products

$$\Delta(H_{1N}) = H_{1N} \otimes T + T^{-1} \otimes H_{1N}, \qquad \Delta(T^{\pm 1}) = T^{\pm 1} \otimes T^{\pm 1},$$

$$\Delta(E_{N1}) = E_{N1} \otimes T + T^{-1} \otimes E_{N1},$$

$$\Delta(H_{ij}) = H_{ij} \otimes 1 + 1 \otimes H_{ij} - \frac{1}{4}(\delta_{i1} + \delta_{jN})(T H_{1N} \otimes (1 - T^2) + (1 - T^{-2}) \otimes T^{-1} H_{1N}),$$
$$i < j \in \{1, 2, \ldots, N\} \quad and \quad (i, j) \neq (1, N),$$

$$\Delta(E_{ij}) = E_{ij} \otimes T^{-(\delta_{i1}+\delta_{jN})/2} + T^{(\delta_{i1}+\delta_{jN})/2} \otimes E_{ij},$$
$$i < j \in \{1, 2, \ldots, N\} \quad and \quad (i, j) \neq (1, N),$$

$$\Delta(E_{ji}) = E_{ji} \otimes T^{(\delta_{i1}+\delta_{jN})/2} + T^{-(\delta_{i1}+\delta_{jN})/2} \otimes E_{ji} + \frac{\mathsf{h}}{4} T^{-1}(-\delta_{i1} E_{jN} + \delta_{jN} E_{1i}) \otimes (T^{-1/2} H_{1N}$$
$$+ H_{1N} T^{-1/2}) - \frac{\mathsf{h}}{4}(T^{1/2} H_{1N} + H_{1N} T^{1/2}) \otimes T(-\delta_{i1} E_{jN} + \delta_{jN} E_{1i})$$
$$i < j \in \{1, 2, \ldots, N\} \quad and \quad (i, j) \neq (1, N),$$

$$\Delta(H_{i,N+1}) = H_{i,N+1} \otimes 1 + 1 \otimes H_{i,N+1} - \frac{1}{4}(\delta_{i1} - \delta_{iN})(T H_{1N} \otimes (1 - T^2)$$
$$+ (1 - T^{-2}) \otimes T^{-1} H_{1N}), \qquad i \in \{1, 2, \ldots, N\},$$

$$\Delta(E_{i,N+1}) = E_{i,N+1} \otimes T^{(-\delta_{i1}+\delta_{iN})/2} + T^{(\delta_{i1}-\delta_{iN})/2} \otimes E_{ji} + \frac{\mathsf{h}}{4} T^{-1} \delta_{iN} E_{1N} \otimes (T^{-1/2} H_{1N}$$
$$+ H_{1N} T^{-1/2}) - \frac{\mathsf{h}}{4}(T^{1/2} H_{1N} + H_{1N} T^{1/2}) \otimes \delta_{iN} T E_{1N} \qquad i \in \{1, 2, \ldots, N\},$$

$$\Delta(E_{N+1,i}) = E_{N+1,i} \otimes T^{(\delta_{i1}-\delta_{iN})/2} + T^{(-\delta_{i1}+\delta_{iN})/2} \otimes E_{N+1,i} - \frac{\mathsf{h}}{4} T^{-1} \delta_{i1} E_{N+1,N} \otimes (T^{-1/2} H_{1N}$$
$$+ H_{1N} T^{-1/2}) + \frac{\mathsf{h}}{4}(T^{1/2} H_{1N} + H_{1N} T^{1/2}) \otimes \delta_{i1} T E_{N+1,N} \quad i \in \{1, 2, \ldots, N\}. \tag{4.2}$$

The commutator rules of $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ can be evaluated by direct calculations.

Parallelling the earlier cases, the universal $\mathcal{R}_{\mathsf{h}}$-matrix of $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ is given by

$$\mathcal{R}_{\mathsf{h}} = \exp(-\mathsf{h} E_{1N} \otimes T H_{1N}) \exp(\mathsf{h} T H_{1N} \otimes E_{1N}), \tag{4.3}$$

where $E_{1N} = \mathsf{h}^{-1} \ln T$. This element can be connected to the results obtained by the contraction process by a suitable twist operator that can be derived as a series expansion in h.

5. Conclusion

In general, a class of nonlinear maps exists relating the Jordanian quantum (super)algebras and their classical analogues. Here we have used a particular map realizing Jordanian $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ super-algebra for an arbitrary N. This map arises naturally form our contraction process defined in (1.1). Let us recall that the more important advantages of our procedure are:

- The algebraic commutation relations are deformed.
- The co-algebraic structure is simpler.
- The map obtained, by our contraction process, permits immediate explicit construction of the finite-dimensional irreps.
- Ohn's $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra is embedded as a Hopf subalgebra in our construction. Therefore, the Jordanian $\mathcal{U}_{\mathsf{h}}(sl(N|1))$ super-algebra arising from our method corresponds to the classical r-matrix $r = h_{1N} \otimes e_{1N} - e_{1N} \otimes h_{1N}$.

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